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Linear-Logic Based Analysis of Constraint Handling Rules

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Classical Logic I

What is Logic?

- “The study of correct reasoning, especially as it involves the drawing of inferences.” (Encyclopaedia Britannica)
- “The anatomy of thought.” (John Locke)
- “The art of going wrong with confidence.” (Joseph Wood Krutch)

Classical Logic II

Logical Symbols

- Connectives: $\wedge \quad \rightarrow \quad \vee \quad \neg$
- Consequence: $\models$
- Quantifiers: $\exists \quad \forall$

Examples

- $\mathbf{rain} \wedge \mathbf{sun} \rightarrow \mathbf{rainbow}$
- $\models \mathbf{rain} \vee \neg\mathbf{rain}$
- $\mathbf{rain} \rightarrow \mathbf{wet}, \mathbf{rain} \models \mathbf{wet}$
- $\mathbf{rain} \rightarrow \mathbf{wet}, \mathbf{rain} \models \mathbf{rain} \wedge \mathbf{wet}$
- $\forall \mathbf{D}.\mathbf{rain}(\mathbf{D}) \rightarrow \mathbf{wet}(\mathbf{D}), \mathbf{rain}(\mathbf{today}) \models \mathbf{wet}(\mathbf{today})$

Classical Logic III

Semantical vs. Syntactical Consequence

- $\varphi_1, \ldots, \varphi_n \models \psi$
  - semantical consequence
  - model-theoretic characterization
- $\varphi_1, \ldots, \varphi_n \vdash \psi$
  - syntactical consequence
  - proof-theoretic characterization

Non-Classical Logics

- *extensions* of CL /*deviations* from CL
- truth value/interpretation/model?

Linear Logic I

(Multiplicative) Conjunction)

CL

$A, B \models A \wedge B$

$A, A \models A \wedge A$

$A \equiv A \wedge A \quad \begin{cases} A \vdash A \wedge A \\ A \wedge A \vdash A \end{cases}$

set semantics
truths

LL

$A, B \vdash A \otimes B$

$A, A \vdash A \otimes A$

$A \not\equiv A \otimes A \quad \begin{cases} A \nvdash A \otimes A \\ A \otimes A \nvdash A \end{cases}$

multi-set semantics
resources

Example

- **rain** $\wedge$ **rain** $\equiv$ **rain**
- **coffee** $\otimes$ **coffee** $\not\equiv$ **coffee**

Linear Logic II

(Linear) Implication

CL	**LL**
$A, A \rightarrow B \models B$	$A, A \multimap B \vdash B$
$A, A \rightarrow B \models A \wedge B$	$A, A \multimap B \nvdash A \otimes B$
strictly monotonic consequence	“consumes” precondition state transition

Examples

- $\mathbf{rain} \rightarrow \mathbf{wet}, \mathbf{rain} \models \mathbf{rain} \wedge \mathbf{wet}$
- $\mathbf{euro} \rightarrow \mathbf{coffee}, \mathbf{euro} \vdash \mathbf{coffee}$
- $\mathbf{euro} \rightarrow \mathbf{coffee}, \mathbf{euro} \nvdash \mathbf{euro} \otimes \mathbf{coffee}$

Linear Logic III

(Additive) Conjunction

CL	**LL**	
$A \wedge B \models A$	$A \otimes B \nvdash A$	$A \& B \vdash A$
$A \wedge B \models B$	$A \otimes B \nvdash B$	$A \& B \vdash B$
$A, B \vdash A \wedge B$	$A, B \vdash A \otimes B$	$A, B \nvdash A \& B$

Examples

- **coffee**$\&$**pie** $\vdash$ **coffee**
- **coffee**$\&$**pie** $\vdash$ **pie**
- **coffee**$\&$**pie** $\nvdash$ **coffee** $\otimes$ **pie**

Linear Logic IV

"Bang" exponential

CL	**LL**	
$A \equiv A \wedge A$	$A \not\equiv A \otimes A$	$!A \equiv !A \otimes !A$
	$A \otimes B \not\equiv A \& B$	$!A \otimes !B \equiv !(A \& B)$

restores set-semantics
unifies $\otimes$ and &

Examples

- **rain** $\rightarrow$ **wet**, **rain** $\models$ **rain** $\wedge$ **wet**
- **rain** $\multimap$ **wet**, **rain** $\nvdash$ **rain** $\otimes$ **wet**
- $!($**rain** $\multimap$ **wet**$)$, $!$**rain** $\vdash !$**rain**$\otimes !$**wet**
- $!($**euro** $\multimap$ **coffee**&**pie**$)$ $\vdash$ **euro** $\otimes$ **euro** $\multimap$ **coffee** $\otimes$ **pie**

Linear Logic V: Embedding of Classical Logic

Consequences of "bang"

- restores properties of CL
- selective recovery or full embedding

Possible Interpretation

unbanged formula	resource
banged formula	unlimited resource / proposition
$\vdash$	state transition?
	logical consequence?
	$\Rightarrow$ **aspects of both**

Linear Logic VI: First-order Linear Logic

FOLL by Example

- All pies are one euro:
- $!\forall P.!\mathbf{is\_pie}(P) \multimap (\mathbf{euro} \multimap \mathbf{pie}(P))$
- Some pies are one euro:
- $!\exists P.!\mathbf{is\_pie}(P) \otimes (\mathbf{euro} \multimap \mathbf{pie}(P))$

Translation of States

Translation of States by Example

$$\langle \mathbf{a}, \mathbf{a}; \top; \emptyset \rangle^L ::= \mathbf{a} \otimes \mathbf{a}$$
$$\langle \mathbf{c}(X); X > 0; \{X\} \rangle^L ::= \mathbf{c}(X) \otimes !(X > 0)$$
$$\langle \mathbf{c}(X); Y > 0; \{X\} \rangle^L ::= \exists Y.\mathbf{c}(X) \otimes !(Y > 0)$$

Translation of States

user-defined constraints	unbanged atoms
built-in constraints	banged atoms
global variables	free variables
local variables	ex. quantified variables

Translation of Rules

Translation of Rules by Example

$\mathbf{a} \Leftrightarrow \mathbf{b}$ $\qquad \langle \mathbf{a}; \top; \emptyset \rangle \mapsto \langle \mathbf{b}; \top; \emptyset \rangle$

$!(\mathbf{a} \multimap \mathbf{b}) \vdash \mathbf{a} \multimap \mathbf{b}$

$\mathbf{a}(X) \Leftrightarrow \mathbf{b}(X)$ $\qquad \langle \mathbf{a}(0); \top; \emptyset \rangle \mapsto \langle \mathbf{b}(0); \top; \emptyset \rangle$

$!\forall(\mathbf{a}(X) \multimap \mathbf{b}(X)) \vdash \mathbf{a}(0) \multimap \mathbf{b}(0)$

$\mathbf{a}(X) \Leftrightarrow X \geq 0 \mid \mathbf{b}(X)$ $\qquad \langle \mathbf{a}(0); \top; \emptyset \rangle \mapsto \langle \mathbf{b}(0); \top; \emptyset \rangle$

$!\forall(!(X \geq 0) \multimap (\mathbf{a}(X) \multimap \mathbf{b}(X))) \vdash \mathbf{a}(0) \multimap \mathbf{b}(0)$

$\mathbf{a} \Rightarrow \mathbf{b}$ $\qquad \langle \mathbf{a}; \top; \emptyset \rangle \mapsto \langle \mathbf{a}, \mathbf{b}; \top; \emptyset \rangle$

$!(\mathbf{a} \multimap \mathbf{a} \otimes \mathbf{b}) \vdash \mathbf{a} \multimap \mathbf{a} \otimes \mathbf{b}$

Soundness and Completeness of Linear Logic Semantics

Soundness

$$P, \mathcal{CT} : S \mapsto^* T \quad \Rightarrow \quad P^L, \mathcal{CT}^L \vdash S^L \multimap T^L$$

No Completeness?

$$P^L, \mathcal{CT}^L \vdash S^L \multimap T^L \quad \not\Rightarrow \quad P, \mathcal{CT} : S \mapsto^* T$$

Implicit Weakening in the Completeness Result

Example: Implicit Weakening

Let $P = \{a(X) \Leftrightarrow b(X)\}$.

$$P, \mathcal{CT} : \langle a(0); \top; \emptyset \rangle \mapsto \langle b(0); \top; \emptyset \rangle \quad (1)$$

$$P^L, \mathcal{CT}^L \vdash a(0) \multimap b(0) \quad (2)$$

$$\vdash b(0) \multimap \exists X.b(X) \quad (3)$$

$$P^L, \mathcal{CT}^L \vdash a(0) \multimap \exists X.b(X) \quad (4)$$

$$P, \mathcal{CT} : \langle a(0); \top; \emptyset \rangle \not\mapsto \langle b(X); \top; \emptyset \rangle \quad (5)$$

Theorem (Completeness)

If $P^L, \mathcal{CT}^L \vdash S^L \multimap T^L$ then

$P, \mathcal{CT} : S \mapsto^*_P U$ and $\mathcal{CT}^L \vdash U^L \multimap T^L$.

Linear Logic and State Equivalence

Implicit State Equivalence

$$\vdash \langle \mathbf{u}(\mathbf{0}); \top; \{X\}\rangle^L \multimap\!\circ \langle \mathbf{u}(\mathbf{0}); \top; \emptyset\rangle^L$$

$$\mathcal{CT}^L \vdash \langle \mathbf{u}(\mathbf{X}); X = 0; \{X\}\rangle^L \multimap\!\circ \langle \mathbf{u}(\mathbf{0}); X = 0; \{X\}\rangle^L$$

$$\mathcal{CT}^L \vdash \langle \mathbb{U}; \bot; \mathbb{V}\rangle^L \multimap\!\circ \langle \mathbb{U}'; \bot; \mathbb{V}'\rangle^L$$

Equivalence of States (Preliminary Definition)

$$S \equiv T \quad \Leftrightarrow \quad \mathcal{CT}^L \vdash S^L \multimap\!\circ T^L$$

Axiomatic Definition of Equivalence

State Equivalence

Let state equivalence be the smallest equivalence relation $\equiv_e$ over states such that:

1. $\langle \mathbb{U}; X = t \wedge \mathbb{B}; \mathbb{V} \rangle \equiv_e \langle \mathbb{U}\,[X/t]\,; X = t \wedge \mathbb{B}; \mathbb{V} \rangle$
2. Let $\bar{s}_i = vars(\mathbb{B}_i) \setminus vars(\mathbb{U}, \mathbb{V})$. If $\mathcal{CT} \models \exists \bar{s}_1.\mathbb{B}_1 \leftrightarrow \exists \bar{s}_2.\mathbb{B}_2$ then
$$\langle \mathbb{U}; \mathbb{B}_1; \mathbb{V} \rangle \equiv_e \langle \mathbb{U}; \mathbb{B}_2; \mathbb{V} \rangle$$
3. For $X \notin vars(\mathbb{U}, \mathbb{B})$, $\langle \mathbb{U}; \mathbb{B}; \{X\} \cup \mathbb{V} \rangle \equiv_e \langle \mathbb{U}; \mathbb{B}; \mathbb{V} \rangle$
4. $\langle \mathbb{U}; \bot; \mathbb{V} \rangle \equiv_e \langle \mathbb{U}'; \bot; \mathbb{V}' \rangle$

Coincidence of Equivalence Definitions

Theorem (Coincidence of Definitions)

The *axiomatic* definition of state equivalence coincides with *implicit* state equivalence:

$$S \equiv_e T \quad \Leftrightarrow \quad \mathcal{CT}^L \vdash S^L \multimap\!\circ\; T^L$$

Safety Properties

Safety Properties

Any property of the form $P, \mathcal{CT} : S \not\mapsto T$ is called a *safety property*.

Sufficient Criterion for Safety Properties

$$P^L, \mathcal{CT}^L \nvdash S^L \multimap T^L \quad \Rightarrow \quad P, \mathcal{CT} : S \not\mapsto^* T$$

Operational Equivalence

Definition (Operational $\mathcal{S}$-Equivalence)

Two CHR programs P_1, P_2 are *operationally $\mathcal{S}$-equivalent* if for any two states S and $\langle \emptyset; \mathbb{B}; \mathbb{V} \rangle$, we have:

$$P_1, \mathcal{CT} : S \mapsto^* \langle \emptyset; \mathbb{B}; \mathbb{V} \rangle \quad \Leftrightarrow \quad P_2, \mathcal{CT} : S \mapsto^* \langle \emptyset; \mathbb{B}; \mathbb{V} \rangle$$

Sufficient Criterion for Operational $\mathcal{S}$-Equivalence

Definition: Confluence

A CHR program P is *confluent* if for all states S, T, T' such that $S \mapsto^* T$ and $S \mapsto^* T'$, there exists a state T'' such that $T \mapsto^* T''$ and $T' \mapsto^* T''$.

Logical equivalence is *sufficient* for $\mathcal{S}$-equivalence:

Theorem: $\mathcal{S}$-Equivalence

Let P_1, P_2 be *confluent* CHR programs such that:

$$\mathcal{CT}^L \vdash P_1^L \multimap\!\circ\, P_2^L$$

Then P_1, P_2 are $\mathcal{S}$-equivalent.

Outlook

Further Applications

- Extension to CHR with search ($CHR^{\vee}$)
  - Embedding LP programs
  - Deciding operational equivalence across language paradigms
- Novel ways to deal with propagation
  - Trivial non-termination in naive semantics
  - Inspired by "bang" exponential:
  - Finite representation of infinite states

Summary

- Linear Logic ...
  - ...deals with resources *and* truths
  - ...is non-monotonic
  - ...embeds classical logic
- CHR ...
  - ...corresponds to a subset of linear logic
  - ...can be analysed using linear logic
- Applications include ...
  - ...motivation and justification for state equivalence
  - ...checking safety properties
  - ...deciding operational equivalence
  - ...even across language paradigms